$\lambda\phi^4$ Model

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Model Formulation

$\lambda\phi^4$ model describes the scalar field $\phi(x)$ interacting with itself. The action reads

$$S[\phi] = S_0[\phi] + S_I[\phi] = \frac{1}{2}\,\phi\, L\, \phi + \frac{\lambda}{4!}\,\phi^4, \tag{1}$$

where $L = -\partial^2 - m^2$ is the free field Lagragain density operator. Here we have adopted an abbriviation by omitting the integral over spacetime. The full form should be

$$S_0[\phi] = \frac{1}{2}\,\phi\, L\, \phi = \frac{1}{2}\int d^4 x\, \phi\,(x)\left(-\partial^2 - m^2\right)\phi\,(x), \tag{2}$$

$$S_I[\phi] = \frac{\lambda}{4!}\,\phi^4 = \frac{\lambda}{4!}\int d^4 \eta\, \phi^4\,(\eta). \tag{3}$$

The goal is to evaluate the partition function with source term, which is just like taking a Fourier transform of $e^{i\, S[\phi]}$,

$$\boxed{Z_\lambda[J] = \int \mathcal{D}\phi\, e^{i\,(S[\phi] + \phi\, J)} = \int \mathcal{D}\phi\, e^{i\left(\frac{1}{2}\phi\, L\, \phi + \frac{\lambda}{4!}\phi^4 + \phi\, J\right)}.} \tag{4}$$

The term $\phi\, J$ is also an abbriviated notation, which should be understood as $\int d^4\xi\, \phi(\xi)\, J(\xi)$.

Structure of Partition Function

There is no hope to evaluate the whole partition function $Z_\lambda[J]$ directly. $Z_\lambda[J]$ is always given in the power series of λ and J

$$Z_\lambda[J] = \sum_{m,n} z_{m\,n}\,(i\,\lambda)^m\,(i\,J)^n. \tag{5}$$

Again the notation $z_{m\,n}(i\,J)^n$ is sort for

$$z_{m\,n}\,(i\,J)^n = \int d^4 x_1 \ldots \int d^4 x_n\, z_{m\,n}\,(x_1, \ldots, x_n)\, i\, J\,(x_1) \ldots i\, J\,(x_n), \tag{6}$$

where the z-coefficients are formally given by

$$z_{m\,n}\,(x_1, \ldots, x_n) = \frac{1}{m!\,n!}\left(\frac{\delta}{i\,\delta\, J\,(x_1)} \cdots \frac{\delta}{i\,\delta\, J\,(x_n)}\,\frac{\partial^m Z_\lambda[J]}{\partial(i\,\lambda)^m}\right)_{\lambda\to 0,\, J\to 0}. \tag{7}$$

One gains full information about $Z_\lambda[J]$, if all the z-coefficients are obtained. There are two approaches to evaluate the z-coefficients: Schwinger's approach and Wick's approach.

Schwinger's Approach

The key idea of Schwinger's approach is to make use of the functional derivative in the dual space, since $Z_\lambda[J]$ is understood as a Fourier transformation from ϕ to J (the dual field of ϕ).

General Idea

Make a substitution $\phi(x) \to \delta/\delta J(x)$ in $e^{i S_I[\phi]}$, and move it out of the path integral.

$$Z_\lambda[J] = \exp\left(\frac{i\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4\right)\int \mathcal{D}\phi\, e^{i\left(\frac{1}{2}\phi L \phi + \phi J\right)}, \tag{8}$$

where the notation $(\delta/\delta J)^4$ is an abbriviation for

$$\left(\frac{\delta}{\delta J}\right)^4 = \int d^4\eta \left(\frac{\delta}{\delta J(\eta)}\right)^4. \tag{9}$$

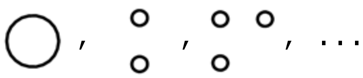
The path integral left over in Eq. (8) is of Gaussian type, which can be carried out by completing the square

$$\frac{1}{2}\phi L \phi + \phi J = \frac{1}{2}\left(\phi + J L^{-1}\right) L \left(\phi + L^{-1} J\right) - \frac{1}{2} J L^{-1} J. \tag{10}$$

Introducing the propagator $G = L^{-1}$, the result can be written as

$$\boxed{Z_0[J] = \int \mathcal{D}\phi\, e^{i\left(\frac{1}{2}\phi L \phi + \phi J\right)} = Z_0[0]\, e^{\frac{i}{2} J G J}.} \tag{11}$$

$Z_0[0]$ is the vacuum amplitude of the free field, which only contains vacuum bubbles: creation and annihilation of virtual particles.



All these bubbles serve as the background, and can be totally neglected by normalizing the amplitude $Z_0[0]$ to unity.

Up to now we have arrived at the following formula,

$$\boxed{Z_\lambda[J] = \exp\left(\frac{i\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4\right) Z_0[J],} \tag{12}$$

with $Z_0[J]$ given in Eq. (11). If the coupling λ is weak, it can be treated as a perturbation. The spirit of perturbation theory is to power expand with respect to the perturbations

$$Z_\lambda[J] = \left(1 + \frac{i\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4 + \frac{1}{2!}\left(\frac{i\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4\right)^2 + \ldots\right) Z_0[J]. \tag{13}$$

Now we have the partition function in λ series, from which the mth order partial derivative of $Z_\lambda[J]$

with respect to $i\lambda$ can be extracted from the coefficient in front of $(i\lambda)^m$,

$$\left(\frac{\partial^m Z_\lambda[J]}{\partial(i\lambda)^m}\right)_{\lambda\to 0} = \frac{1}{(4!)^m}\left(\left(\frac{\delta}{\delta J}\right)^4\right)^m Z_0[J]. \tag{14}$$

Substitute into Eq. (7), we see the remaining job is to finish all the functional derivatives $\delta/\delta J$. Then the z-coefficient can be calculated one by one to the desired order.

Functional Derivative

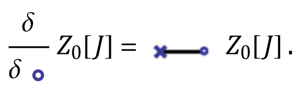
The most important step towards pratical use of Schwinger's approach is to derive the following rule of functional derivative

$$\frac{\delta}{i\,\delta J} Z_0[J] = Z_0[0]\,\frac{\delta}{i\,\delta J}\,e^{\frac{i}{2}JGJ} = Z_0[0]\,e^{\frac{i}{2}JGJ}\,\frac{\delta}{i\,\delta J}\left(\frac{i}{2}JGJ\right) = G\,J\,Z_0[J], \tag{15}$$

or more seriously written as

$$\frac{\delta}{i\,\delta J(x)} Z_0[J] = \left(\int d^4\xi\, G(x,\xi)\, J(\xi)\right) Z_0[J]. \tag{16}$$

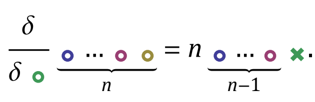
This has a graphical representation. If we denote the source $i\,J(\xi)$ by a little circle and the propagator $-i\,G(x,\xi)$ by a line , then the integral in Eq. (16) could be represented by the diagram , where the cross  is to anchor one variable of the propagator at the spacetime position x. The little circle is free to move around, indecating $i\,J(\xi)$ should be integrate over the spacetime; while the cross  is fixed and should not be involved in the integration. The formula Eq. (16) can be concluded as


(17)

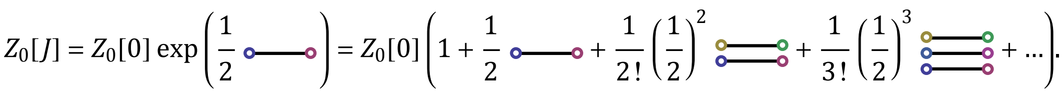
Another simple but also important rule: the functional derivative anchors the source.

$$\frac{\delta}{i\,\delta J(x)}(i\,J)^n = n\,(i\,J)^{n-1}\,\frac{\delta J(y)}{\delta J(x)} = n\,(i\,J)^{n-1}\,\delta(x-y). \tag{18}$$

Note the multiple n in the result. Let us draw a picture to present it.


(19)

With this rule, one can varify Eq. (17), given the graphical representation of $Z_0[J]$,


(20)

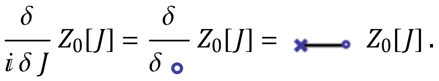
Repeatedly applying the rules of Eq. (17) and Eq. (19), we are able to do the functional derivative now.

First Oder Expansion

Now we get down to evaluate the partition function to the first order of λ by Schwinger's approach. Starting from Eq. (13), take the first two terms

$$Z_\lambda[J] = Z_0[J] + \frac{i\lambda}{4!}\left(\frac{\delta}{i\,\delta J}\right)^4 Z_0[J]. \tag{21}$$

The second term contains four times of functional derivatives which should be carried out one by one.



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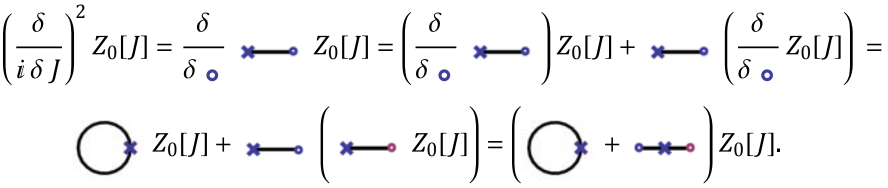
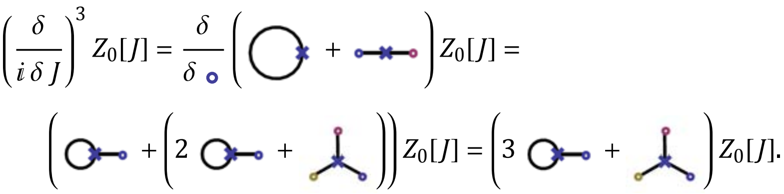


(23)

Note that the same crosses × are put together automatically, because all the four direvatives $\delta/\delta J(x)$ are taken at the same spacetime point x.



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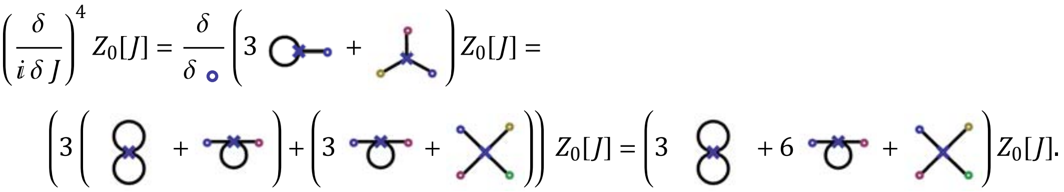
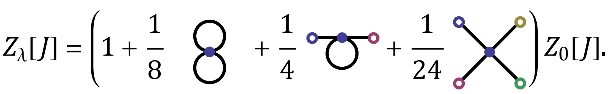


(25)

The final step is to substitude the result into Eq. (21). The cross × will be replaced by a dot • (representing $i\lambda$) with the factor $1/4!$.



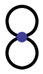
(26)

We have obtain the expansion of partition function to the first order of $i\lambda$. Refering to the following table, one can translate the diagrams back to regular mathematical expressions.

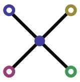
[× $= \delta(\xi - x)$],
• $= i\lambda$,
∘ $= i\,J(\xi)$,
—— $= -i\,G(\xi_1, \xi_2)$.

(27)

Move the mouse to the links and vertices in each diagram to see the corresponding interpretation.

 $= -i\,\lambda \int G(\eta_1, \eta_1)\, G(\eta_1, \eta_1)\, d^4\eta_1,$

 $= \lambda \int G(\eta_1, \eta_1)\, G(\eta_1, \xi_1)\, G(\eta_1, \xi_2)\, J(\xi_1)\, J(\xi_2)\, d^4\eta_1\, d^4\xi_1\, d^4\xi_2,$

 $=$

$$i\,\lambda \int G(\eta_1, \xi_1)\, G(\eta_1, \xi_2)\, G(\eta_1, \xi_3)\, G(\eta_1, \xi_4)\, J(\xi_1)\, J(\xi_2)\, J(\xi_3)\, J(\xi_4)\, d^4\eta_1\, d^4\xi_1\, d^4\xi_2\, d^4\xi_3\, d^4\xi_4.$$

These diagrams are known as Feynman diagrams, and the rules in Eq. (27) are Feynman rules.

From the expansion in Eq. (26), we can at least determine the following four z-coefficients

$$z_{00} = Z_0[0], \tag{29}$$

$$z_{10} = -\frac{Z_0[0]}{8} \int G(\eta_1, \eta_1)\, G(\eta_1, \eta_1)\, d^4\eta_1, \tag{30}$$

$$z_{12}(x_1, x_2) = \frac{i\,Z_0[0]}{4} \int G(\eta_1, \eta_1)\, G(\eta_1, x_1)\, G(\eta_1, x_2)\, d^4\eta_1, \tag{31}$$

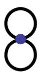
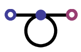
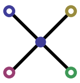
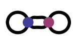
$$z_{14}(x_1, x_2, x_3, x_4) = \frac{Z_0[0]}{24} \int G(\eta_1, x_1)\, G(\eta_1, x_2)\, G(\eta_1, x_3)\, G(\eta_1, x_4)\, d^4\eta_1. \tag{32}$$

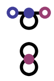
The z-factors (expect z_{00}) are integrals of some poduct of the propagators.

Exersice: Check the Expansion to the Second Order

Write a computer program to calculate the $i\lambda$ expansion automatically. Check that the partition function to the second order of $i\lambda$ reads

$Z_\lambda[J] =$

$\Big(1 + \frac{1}{8}$  $+ \frac{1}{4}$  $+ \frac{1}{24}$  $+ \frac{1}{128}$  $+ \frac{1}{16}$  $+ \frac{1}{48}$  $+ \frac{1}{8}$  $+$

$\frac{1}{32}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \frac{1}{32}$  $+ \frac{1}{16}$  $+$ (33)

$\frac{1}{12}$  $+ \frac{1}{192}$  $+ \frac{1}{72}$  $+ \frac{1}{96}$  $+ \frac{1}{1152}$  $\Big) Z_0[J].$

Can you determine the factors in the front just by looking at the diagram?

Programs

Symmetry Factor

We have noticed that in front of each diagram there is a factor. For example there is a $1/8$ in front of the diagram . This $1/8$ is known as the symmetry factor of , which can be determined from the topological symmetry of the diagram directly. The rules are concluded as follows:

Rule (1) Each **circle**  contributes a factor $1/2$.

Rule (2) **Multiple links**  between two vertecies contributes a factor $1/n!$, where n is the number of links.

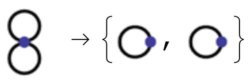
Rule (3) Diagram with ***n*-fold** permutative **symmetry** should be attached to by a factor $1/n!$. (e.g. the symmetry factor of is $1/2!$ and that of  is $1/5!$).

Rule (4) If the diagram consists of n **identical sub-diagrams**, each of which has a symmetry factor s, then the symmetry factor of the diagram is $s^n/n!$.

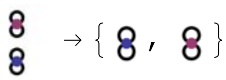
Rule (5) The symmtry factor contributed from each distinct sub-diagram should be multiplied together.

Examples

Use these rules to deduce the symmetry factor of

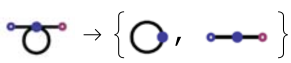


which has two identical circles, satisfying the case of rule (4). Each **circle** contributes a symmetry factor $1/2$, so $s = 1/2$. There are **two** such circles, so $n = 2$. Use the formula $s^n/n! = 1/(2^2 \times 2!) = 1/8$. Therefore the factor in front of this diagram is $1/8$. The same rule also explains the symmetry factor of



which contains two ($n = 2$) identical  (color does not make things distinct topologically), each with a factor $s = 1/8$. So its symmetry factor is $s^n/n! = 1/(8^2 \times 2!) = 1/128$.

Look at the diagram

.

One circle contributes $1/2$. The sub-diagram also contributes $1/2$, because it as a 2-fold symmetry under permutation of the blue and red little circle, hence a symmetry factor $1/2! = 1/2$ according to rule (3). Finally according to rule (5), put the symmetry factors together, we have $1/2 \times 1/2 = 1/4$ for this diagram.

As an example for rule (2), look at the diagram

which as 4 links between the vertices and . Therefore the multiple links contribute a symmetry factor 1/4!. Also notice the 2-fold symmetry under permutation of the identical vertices and , which will further brings a symmetry factor 1/2!. So the symmetry factor of this diagram is 1/4!×1/2! = 1/48.

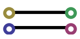
Exersice: determine the symmetry factor of each diagram in Eq. (33).

Deduction of Symmetry Factor

Programs

Connected and Disconnected

Disconnected diagrams are those diagrams that can be partitioned into sub-diagrams without breaking any links. This means the amplitude of a disconnected diagram can be given as the product of the amplitudes of connected diagrams. For example,



$$= -\int G(\xi_1, \xi_2)\, G(\xi_3, \xi_4)\, J(\xi_1)\, J(\xi_2)\, J(\xi_3)\, J(\xi_4)\, d^4\xi_1\, d^4\xi_2\, d^4\xi_3\, d^4\xi_4 =$$

$$\left(i \int G(\xi_1, \xi_2)\, J(\xi_1)\, J(\xi_2)\, d^4\xi_1\, d^4\xi_2\right)\left(i \int G(\xi_3, \xi_4)\, J(\xi_3)\, J(\xi_4)\, d^4\xi_3\, d^4\xi_4\right) = \left(\right)^2 \tag{35}$$

The symmetry factor just so happens that if appears in the power series expansion with a factor 1/2, then ()^n will appear with the factor $1/n! \times (1/2)^n$. Thus if all these diagrams are summed up, we will obtain a single exponent of the connected diagram (refer to Eq. (20)).

$$\sum_n \frac{1}{n!}\left(\frac{1}{2}\;\square\right)^n = \exp\left(\frac{1}{2}\;\square\right). \tag{36}$$

This give us a hint that the sum of disconnected diagrams can be combined into exponents and collected as a factor from the power series. To concrete our discussion, we look at the expansion of the following exponent

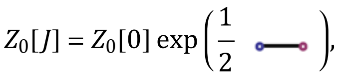
$$\exp(z_1 + z_2 + \ldots) = \sum_n \frac{1}{n!}(z_1 + z_2 + \ldots)^n = \sum_n \frac{1}{n!} \sum_{m_1} \frac{n!}{m_1!\,(n - m_1)!} z_1^{m_1} (z_2 + \ldots)^{n - m_1} =$$

$$\sum_n \sum_{m_1} \frac{z_1^{m_1}}{m_1!} \frac{(z_2 + \ldots)^{n - m_1}}{(n - m_1)!} = \ldots = \sum_{m_1, m_2, \ldots} \frac{z_1^{m_1}}{m_1!} \frac{z_2^{m_2}}{m_2!} \ldots. \tag{37}$$

Now consider z_1, z_2, ... as different connected diagrams (with their symmetry factors attached respectively), then the term

$$\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\ldots.$$

will cover all possible disconnected diagrams made of z_1, z_2, ... with appropriate symmetry factors. Thus we conclude that summing over all disconnected diagrams ends up with a single exponent of the sum of the constituent connected diagrams. Therefore, we are justified to write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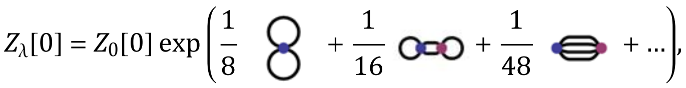
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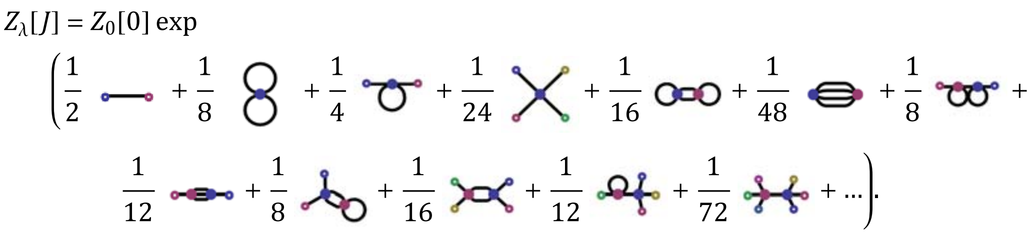
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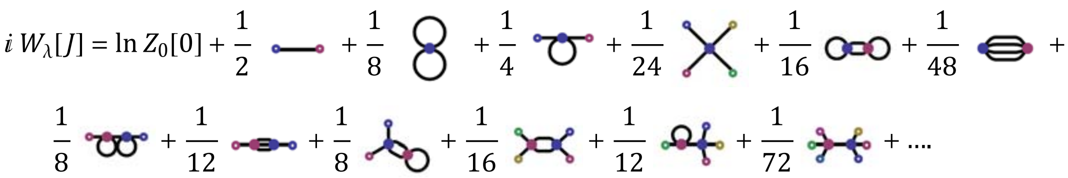
If one introduces the generating functional $W_\lambda[J]$ such that

$$Z_\lambda[J] = e^{i W_\lambda[J]}, \tag{41}$$

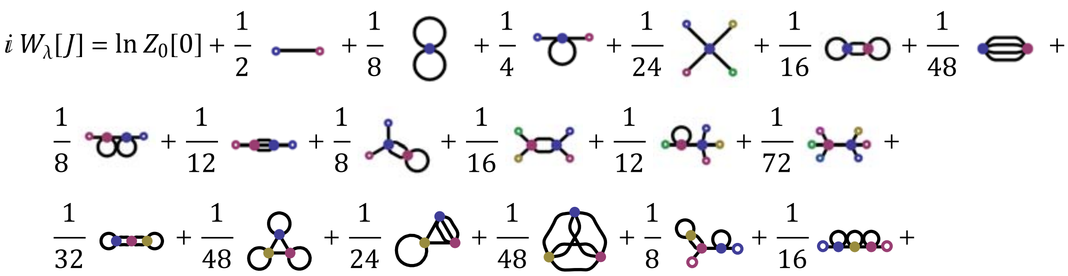
the above result can be more simply written as

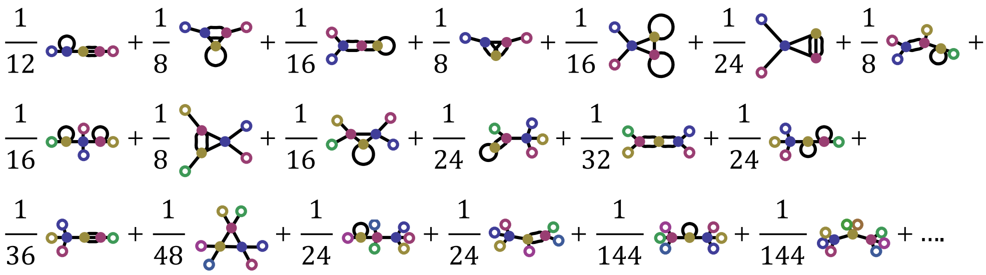

(42)

$W_\lambda[J]$ is in the position of action. It is just like the free energy (times a time-dimension constant), that $\ln Z_0[0]$ is like the zero-point energy, and the rest of diagrams are contribution from the perturbation corrections.

Exersice: Connected Diagrams to $(i\,\lambda)^3$

Show that to the order of $(i\,\lambda)^3$ the generating functional $W_\lambda[J]$ can be given by





Wick's Approach

Script

References

[1] A. Zee, Quantum Field Theory in a Nutshell. Princeton University Press (2003).